Coupled-Mode Theory of Two Nonparallel Dielectric Waveguides

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Abstract—**A general theory for treating the coupling between two nonparallel dielectric waveguides is developed using the coupled-mode assumption. This theory is used to analyze directional couplers consisting of two circularly curved, single-mode dielectric slab waveguides. By assuming continuous coupling between the two waveguides rather than only between adjacent segments on the two waveguides, the present theory avoids the awkwardness of having to specify in a somewhat arbitrary manner the separation between these segments, as is the case for existing theories reported in the literature. It is shown that this over-simplification results often in an overestimate of the power transfer in a directional coupler by 10–20 percent, compared to this theory.**

I. Introduction

THE COUPLING between dielectric waveguides has received much attention [1]–[6] in the last decade in light of the development of fiberoptics and, more recently, integrated circuits in the optical and millimeter-wave bands. Devices needed for future systems such as directional couplers require that the coupling between waveguides be understood so performance can be optimized. Although coupling between parallel guides has been well understood for some time [7], [8], coupling between nonparallel guides is still not clearly understood, even though any physically realizable device must have nonparallel connecting sections where coupling takes place.

Making the coupled-mode assumption, many authors use the results of the parallel case (i.e., the amount of coupling is dependent on guide separation and other parameters describing each guide in isolation), and postulate that, in a nonparallel configuration, there exists a one-to-one correspondence between segments on each guide with some function giving a "separation" to use in the parallel results.

Matsuhara and Watanabe [4] assume this distance is given along a line that intersects the guides at equal angles (Fig. 1(a)). Abouzahra and Lewin [3], for the symmetric case, give the distance separating the guides along a line normal to the plane of symmetry (Fig. 1(b)) while Trinh and Mittra [2] use the length of an arc intersecting both guides at right angles (Fig. 1(c)).

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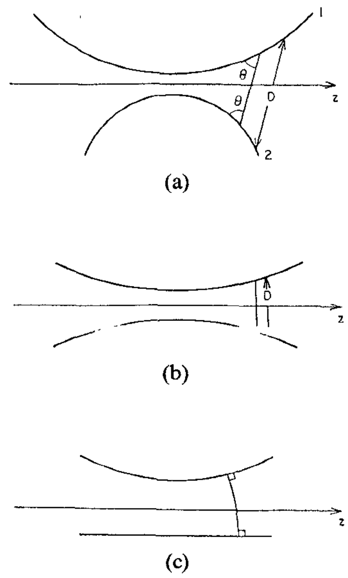

Fig. 1. Various definitions for separation between individual segments of two parallel guides.

By trying to stretch the results of the parallel case, we find not only this problem of determining the "correct" separation, but we overlook the real physical process of interacting time-dependent fields. In the parallel case, any z cross section will have fields with constant phase, while in the nonparallel situation, the phase will not be constant and will change as the configuration on either side of the cross section is changed. Considering only some type of separation between the guides will not completely describe the coupling, because the effect one guide has on another will depend very strongly on the phase, as well as the strength, of the overlapping fields.

In this paper, coupled-mode theory is developed to describe any configuration where the coupled-mode approximation is valid. Approximate coupling coefficients for circularly bent, lossless, single TE-mode slab waveguides are worked out allowing analysis of directional couplers composed of straight and circularly bent slab waveguides.

II. Coupled-Mode Theory of Two Nonparallel Waveguides

The configuration to be analyzed is shown in Fig. 2. The two guides can have different cross sections A_1 and A_2 and dielectric constants ϵ_1 and ϵ_2, but the permeability (μ) is uniform: $\overline{E}_1, \overline{H}_1$ and $\overline{E}_2, \overline{H}_2$ are the two surface-wave

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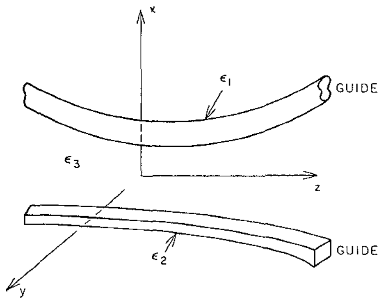

Fig. 2. Two nonparallel dielectric waveguides.

mode fields each guide supports if the other guide is not there; $\tilde{E}$ and $\tilde{H}$ are the time-dependent fields we are interested in for the complete structure consisting of both guides. We then define the functions $\epsilon_1(x, y, z)$ and $\epsilon_2(x, y, z)$ to represent the variation in the relative permittivities when only guide 1 and guide 2, respectively, are presented, i.e.

$$\epsilon_{1,2}(x, y, z) = \begin{cases} \epsilon_{1,2}, & \text{when } \bar{x} \in A_{1,2} \\ \epsilon_3, & \text{otherwise} \end{cases}.$$

Note that $\epsilon_1(x, y, z) = \epsilon_3$ (not ϵ_2) in the region guide 2 occupies and vice versa. $\tilde{\epsilon}(x, y, z)$ is the relative permittivity when both guides exist simultaneously.

The vectors $\bar{F}_1$ and $\bar{F}_2$ can then be defined [1]

$$\bar{F}_{1,2} = \bar{E}_{1,2} \times \tilde{H}^* + \tilde{E}^* \times \bar{H}_{1,2}. \tag{1}$$

By using Maxwell's equations and vector identities, it can be shown that

$$\nabla \cdot \bar{F}_{1,2} = j\omega\epsilon_0(\tilde{\epsilon} - \epsilon_{1,2})\bar{E}_{1,2} \cdot \tilde{E}^*. \tag{2}$$

We make the usual coupled-mode assumption that the total field of a system of two waveguides can be approximated in any given cross section by a linear combination of the individual mode fields

$$\tilde{E} = m_1(z)\bar{E}_1 + m_2(z)\bar{E}_2 \tag{3}$$

$$\tilde{H} = m_1(z)\bar{H}_1 + m_2(z)\bar{H}_2 \tag{4}$$

where m_1 and m_2 are the amplitudes of the two modes at z. Using the vector identity [1] over an infinite cross section A

$$\int_A \nabla \cdot \bar{F} dA = \frac{\partial}{\partial z} \int_A \bar{F} \cdot \hat{z}\, dA \tag{5}$$

we get, from (1) through (5), a set of coupled differential equations for $m_1(z)$ and $m_2(z)$

$$j\left[m_1^*(z)A_{1,2}(z) + m_2^*(z)B_{1,2}(z)\right] = \frac{\partial}{\partial z}\left[m_1^*(z)C_{1,2}(z) + m_2^*(z)D_{1,2}(z)\right] \tag{6}$$

where $A, B, C,$ and D are cross-sectional integrals relating the two individual mode fields

$$A_{1,2}(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\tilde{\epsilon}(x, y, z) - \epsilon_{1,2}(x, y, z))\bar{E}_{1,2} \cdot \bar{E}_1^*\, dx\,dy \tag{7}$$

$$B_{1,2}(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\tilde{\epsilon}(x, y, z) - \epsilon_{1,2}(x, y, z))\bar{E}_{1,2} \cdot \bar{E}_2^*\, dx\,dy \tag{8}$$

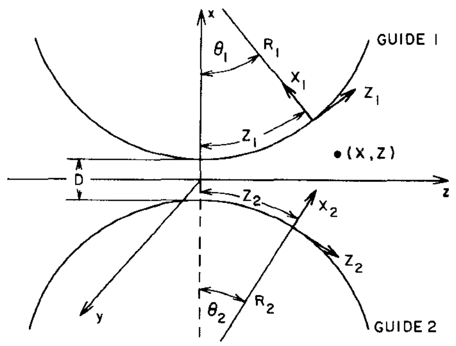

Fig. 3. Global and local coordinate systems.

$$C_{1,2}(z) = \frac{1}{\omega\epsilon_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\bar{E}_{1,2} \times \bar{H}_1^* + \bar{E}_1^* \times \bar{H}_{1,2}\right] \cdot \hat{z}\, dx\,dy \tag{9}$$

$$D_{1,2}(z) = \frac{1}{\omega\epsilon_0}\int_{-\infty}^{\infty}\int\left[\bar{E}_{1,2} \times \bar{H}_2^* + \bar{E}_2^* \times \bar{H}_{1,2}\right] \cdot \hat{z}\, dx\,dy. \tag{10}$$

In general, the coupling coefficients are complex functions of z except for C_1 and D_2, which are real normalization parameters.

We now introduce two locally orthogonal coordinate systems (x_1, z_1) and (x_2, z_2) for the two guides as shown in Fig. 3. If the z dependence of each guide in isolation is $\exp[-j\beta_{1,2}z_{1,2}]$, the coupling coefficients of parallel configurations will have simple $\exp[-j\beta' z]$ dependence (β' = constant, possibly zero). In the nonparallel case, the integration on x at constant z includes a new term of the form $\exp[-j\beta'(z_1 - z_2)]$, where $z_1 - z_2$ now is a function of x. This phase term accounts for the interference of the modal fields which, in the parallel case, does not exist because all phase fronts are parallel.

Theories currently in the literature either assume that the coupling coefficients are the same for the parallel case, except that now the separation is changing in some arbitrary manner [3],[4], or that the amount of transferred power depends on the spacing, as in the parallel case, and define some "equivalent" separation to use at each z cross section [2]. Both approaches neglect field interference in the coupling coefficients which, as shown later, can lead to significant errors.

We note that the use of equivalent current sources [9], or, in the parallel case, the use of variational methods to find the propagation constant of the system modes [8] would appear to give different results. However, we have shown in [10] that the coupled-mode assumption actually leads to identical expressions for these approaches.

III. Analysis of Couplers Formed with Circularly Curved Slab Waveguides

Fig. 3 shows two lossless, single-TE-mode, circularly bent slab waveguides and the relationship between the different coordinate systems. We assume that the radius of curvature of the slabs is large enough so that the field of each bent guide in isolation is approximately that of a straight guide curved about a point. The fields of each guide in isolation are given in Appendix I. To evaluate the coupling coefficients $A_{1,2}$, $B_{1,2}$, $C_{1,2}$, and $D_{1,2}$ in (6), we have to relate the coordinate system for the guides (x_1, z_1) and (x_2, z_2), to the global coordinate system as follows:

$$x_{1,2} = R_{1,2}\left[1-\left\{\left(1+\frac{D}{2R_{1,2}}\mp\frac{x}{R_{1,2}}\right)^2+\left(\frac{z}{R_{1,2}}\right)^2\right\}^{1/2}\right] \tag{11}$$

$$z_{1,2} = R_{1,2}\tan^{-1}\left[\frac{\dfrac{z}{R_{1,2}}}{1+\dfrac{D}{2R_{1,2}}\mp\dfrac{x}{R_{1,2}}}\right] \tag{12}$$

$$\sin\theta_{1,2} = \frac{z}{\left\{\left(R_{1,2}+\dfrac{D}{2}\mp x\right)^2+z^2\right\}^{1/2}} \tag{13}$$

$$\cos\theta_{1,2} = \frac{R_{1,2}+\dfrac{D}{2}\mp x}{\left\{\left(R_{1,2}+\dfrac{D}{2}\mp x\right)^2+z^2\right\}^{1/2}}. \tag{14}$$

Simplifying the expression for the coupling coefficients, we note that the major contribution to the integrals comes from areas near the guides. Also, to avoid radiation losses, the radius of curvature of the bent slab waveguides inherently must be large. These two facts allow us to expand (11)–(14) linearly in x about each of the guides [10] and find analytic solutions for the integrals in (7)–(10) (Appendix III). For example

$$x_1 = \begin{cases} S_{11}(z)+T_{11}(z)\cdot x, & x>0 \text{ (near guide 1)} \\ S_{12}(z)+T_{12}(z)\cdot x, & x<0 \text{ (near guide 2)} \end{cases} \tag{15}$$

where S_{1j} and T_{1j} are given in Appendix II. To solve the remaining coupled differential equations, a second-order Runge Kutta [11] algorithm was used. The approximate coupling coefficients were compared with "exact" numerical integration of these integrals at some selected values in order to establish their accuracy. Seven s on a Cyber 6400 was required for analysis of a configuration carrying all the coupling coefficients.

IV. Numerical Results

Figs. 4 and 5 show the power remaining in guide 1 versus γd (unit power in guide 1 and none in guide 2 initially for an incident power of unity) for the present theory, Abouzahra and Lewin [3] and Trinh and Mittra [2].[1] (Here, d is the separation of the guides including their width: ($d = D+2a$)). Both the symmetric and nonsymmetric cases show that the coupling is substantially smaller when compared with other theories. Thus, the assumption of one-to-one coupling between two corresponding ele-

[1]Comparison with Yariv [12] on a coupler with a long parallel section and negligible coupling in the curved sections shows Trinh and Mittra's k is missing a factor of $1/(1+1/\gamma a)$. With the correction, we let the factor ν be equal to 1 for all cases here.

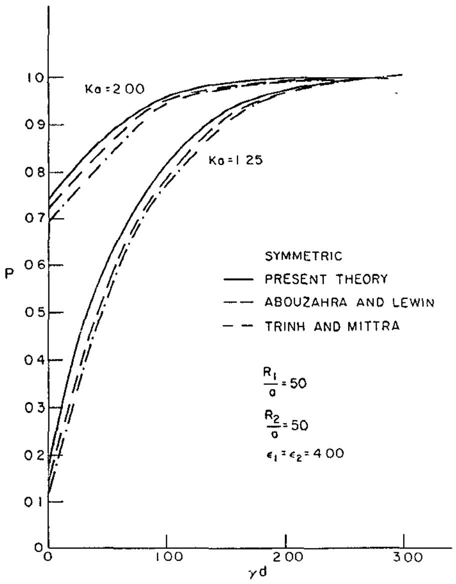

Fig. 4. Final power at guide 1 versus separation.

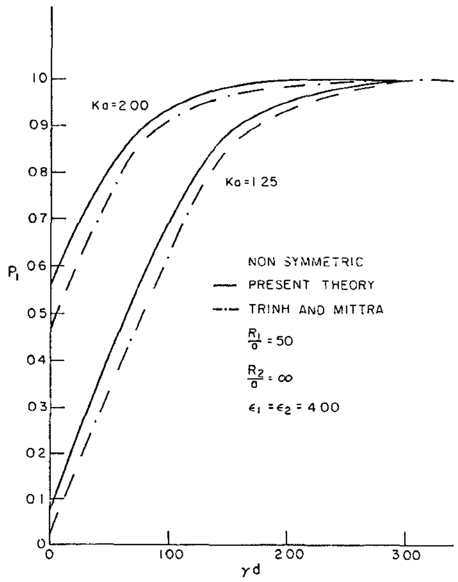

Fig. 5. Final power at guide 1 versus separation.

ments on a pair of nonparallel guides can lead to significant errors when $\gamma d \leqslant 1$. To further demonstrate this point, we have included in Fig. 6 the coupling coefficient B_1 for a symmetrically, curved, degenerate coupler normalized by the constant C_1. In the parallel degenerate case, B_1/C_1 would be a real constant with exponential dependence on guide separation. In all the conventional theories for nonparallel guides which don't account for the phase term, B_1/C_1 would be a real function that exponentially decayed on either side of $z=0$ (minimum guide separation). This type of difference exists also in the other coupling coefficients and leads to the discrepancies of the theories.

It is of interest to note that we found that the total power was not conserved along the coupler for two very closely spaced guides where γd is much smaller than 1, or the physical separation is less than the penetration depth. This likely indicates a breakdown in the coupled-mode

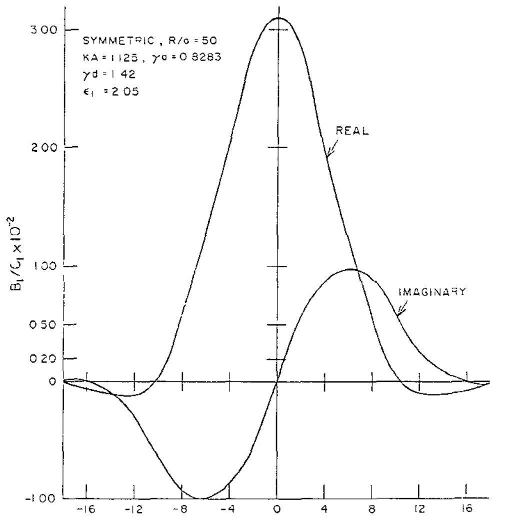

Fig. 6. Coupling coefficient ratio B_1/C_1 versus Z/a.

approximation that includes only one surface-wave mode on each guide. Thus, as it stands, none of the existing theories really is adequate, in designing a circular coupler that provides a complete transfer of power, without the inclusion of a parallel coupling section.

Appendix I
Field of Each Guide in Isolation

Each guide in isolation is a lossless slab structure supporting a single TE-mode. The two guides have the same cross section, the same uniform permittivity and are excited at the same frequency. With these assumptions, the field of either guide in isolation will be [8]

$$E_{y_i} = \cos(px_i)e^{-j\beta z_i}$$

$$H_{z_i} = -\frac{p}{\omega\mu_0}\sin(px_i)e^{-j\beta z_i}, \qquad |x_i| < a \tag{A1}$$

$$H_{x_i} = -\frac{\beta}{\omega\mu_0}\cos(px_i)e^{-j\beta z_i}$$

$$E_{y_i} = \cos(pa)e^{\gamma a}\exp(-\gamma|x_i|)e^{-j\beta z_i}$$

$$H_{z_i} = \frac{j\gamma x_i}{|x_i|\omega\mu_0}\cos(pa)e^{\gamma a}\exp[-\gamma|x_i|]e^{-j\beta z_i}, \qquad |x_i| > a \tag{A2}$$

$$H_{x_i} = -\frac{\beta}{\omega\mu_0}\cos(pa)e^{\gamma a}\exp[-\gamma|x_i|]e^{-j\beta z_i}$$

$$p^2 = k_0^2\epsilon_i - \beta^2$$

$$\gamma^2 = \beta^2 = k_0^2\epsilon_3$$

$$\tan(pa) = \frac{\gamma}{p}. \tag{A3}$$

Appendix II
Linear Approximation of (11)–(14)

$$x_i \cong \begin{cases} S_{11}(z) + T_{11}(z)\cdot x, & x > 0 \text{ (near guide 1)} \\ S_{12}(z) + T_{12}(z)\cdot x, & x < 0 \text{ (near guide 2)} \end{cases} \tag{A4}$$

$$S_{1j} = R_1 - \left\{\left(R_1 + \frac{D}{2} - \Delta_j\right)^2 + z^2\right\}^{1/2} - \frac{\Delta_j\left(R_1 + \frac{D}{2} - \Delta_j\right)}{\left\{\left(R_1 + \frac{D}{2} - \Delta_j\right)^2 + z^2\right\}^{1/2}} \tag{A5}$$

$$T_{1j} = \frac{R_1 + \frac{D}{2} - \Delta_j}{\left\{\left(R_1 + \frac{D}{2} - \Delta_j\right)^2 + z^2\right\}^{1/2}} \tag{A6}$$

$$\Delta_1 = R_1 + \frac{D}{2} - \left\{R_1^2 - z^2\right\}^{1/2} \tag{A7}$$

$$\Delta_2 = -\left(R_2 + \frac{D}{2} - \left\{R_2^2 - z^2\right\}^{1/2}\right) \tag{A8}$$

$$x_2 \cong \begin{cases} S_{21}(z) + T_{21}(z)\cdot x, & x > 0 \\ S_{22}(z) + T_{22}(z)\cdot x, & x < 0 \end{cases} \tag{A9}$$

$$S_{2j} = \left\{\left(R_2 + \frac{D}{2} + \Delta_j\right)^2 + z^2\right\}^{1/2} - \frac{\Delta_j\left(R_2 + \frac{D}{2} + \Delta_j\right)}{\left\{\left(R_2 + \frac{D}{2} + \Delta_j\right)^2 + z^2\right\}^{1/2}} - R_2 \tag{A10}$$

$$T_{2j} = \frac{\left(R_2 + \frac{D}{2} + \Delta_j\right)}{\left\{\left(R_2 + \frac{D}{2} + \Delta_j\right)^2 + z^2\right\}^{1/2}} \tag{A11}$$

$$z_1 - z_2 \cong \begin{cases} V_1(z) + U_1(z)\cdot x, & x > 0 \\ V_2(z) + U_2(z)\cdot x, & x < 0 \end{cases} \tag{A12}$$

$$V_j = \frac{R_1 + R_2}{2}\left[\tan^{-1}(\alpha_{A_j}) - \frac{\Delta_j\cdot\Phi_{A_j}}{1 + \alpha_{A_j}^2}\right] + \frac{R_1 - R_2}{2}\left[\tan^{-1}(\alpha_{B_j}) - \frac{\Delta_j\cdot\phi_{B_j}}{1 + \alpha_{B_j}^2}\right] \tag{A13}$$

$$U_j = \frac{R_1 + R_2}{2}\,\frac{\phi_{A_j}}{1 + \alpha_{A_j}^2} + \frac{R_1 - R_2}{2}\,\frac{\phi_{B_j}}{1 + \alpha_{B_j}^2} \tag{A14}$$

$$\alpha_{A_j} = \frac{z}{\eta_j + z^2}(R_2 - R_1 + 2\Delta_j) \tag{A15}$$

$$\alpha_{B_j} = \frac{z}{\eta_j - z^2}(R_2 + R_1 + D) \tag{A16}$$

$$\eta_j = \left(R_2 + \frac{D}{2} + \Delta_j\right)\left(R_1 + \frac{D}{2} - \Delta_j\right) \tag{A17}$$

$$\phi_{A_j} = \frac{z}{\eta_j + z^2}\left[2 + \frac{(R_2 - R_1 + 2\Delta j)^2}{\eta_j + z^2}\right] \tag{A18}$$

$$\phi_{B_j} = \frac{z}{\left(\eta_j - z^2\right)^2}(R_2 + R_1 + D)(R_2 - R_1 + 2\Delta_j) \tag{A19}$$

$$\sin\Theta_1 \cong \begin{cases} F_{11}(z)+G_{11}(z)\cdot x, & x>0 \\ F_{12}(z)+G_{12}(z)\cdot x, & x<0 \end{cases} \tag{A20}$$

$$F_{1_j} = \frac{z}{\left\{\left(R_1+\frac{D}{2}-\Delta_j\right)^2+z^2\right\}^{1/2}} \cdot\left[1-\frac{\Delta_j\left(R_1+\frac{D}{2}-\Delta_j\right)}{\left(R_1+\frac{D}{2}-\Delta_j\right)^2+z^2}\right] \tag{A21}$$

$$G_{1_j} = \frac{z}{\left\{\left(R_1+\frac{D}{2}-\Delta_j\right)^2+z^2\right\}^{1/2}} \cdot\frac{R_1+\frac{D}{2}-\Delta_j}{\left(R_1+\frac{D}{2}-\Delta_j\right)^2+z^2} \tag{A22}$$

$$\sin\Theta_1 \cong \begin{cases} F_{21}(z)+G_{21}(z)\cdot x, & x>0 \\ F_{22}(z)+G_{22}(z)\cdot x, & x<0 \end{cases} \tag{A23}$$

$$F_{2_j} = \frac{z}{\left\{\left(R_2+\frac{D}{2}+\Delta_j\right)^2+z^2\right\}^{1/2}} \cdot\left[1+\frac{\Delta_j\left(R_2+\frac{D}{2}+\Delta_j\right)}{\left(R_2+\frac{D}{2}+\Delta_j\right)^2+z^2}\right] \tag{A24}$$

$$G_{2_j} = \frac{z}{\left\{\left(R_2+\frac{D}{2}+\Delta_j\right)^2+z^2\right\}^{1/2}} \cdot\frac{-\left(R_2+\frac{D}{2}+\Delta_j\right)}{\left(R_2+\frac{D}{2}+\Delta_j\right)^2+z^2} \tag{A25}$$

$$\cos\Theta_1 \cong \begin{cases} J_{11}(z)+K_{11}(z)\cdot x, & x>0 \\ J_{12}(z)+K_{12}(z)\cdot x, & x<0 \end{cases} \tag{A26}$$

$$J_{1_j} = \frac{1}{\left\{\left(R_1+\frac{D}{2}-\Delta_j\right)^2+z^2\right\}^{1/2}} \cdot\left[R_1+\frac{D}{2}-\frac{\Delta_j\left(R_1+\frac{D}{2}-\Delta_j\right)^2}{\left(R_1+\frac{D}{2}-\Delta_j\right)^2+z^2}\right] \tag{A27}$$

$$K_{1_j} = \frac{1}{\left\{\left(R_1+\frac{D}{2}-\Delta_j\right)^2+z^2\right\}^{1/2}} \cdot\frac{-z^2}{\left(R_1+\frac{D}{2}-\Delta_j\right)^2+z^2} \tag{A28}$$

$$\cos\Theta_2 \cong \begin{cases} J_{21}(z)+K_{21}(z)\cdot x, & x>0 \\ J_{22}(z)+K_{22}(z)\cdot x, & x<0 \end{cases} \tag{A29}$$

$$J_{2_j} = \frac{1}{\left\{\left(R_2+\frac{D}{2}+\Delta_j\right)^2+z^2\right\}^{1/2}} \cdot\left[R_2+\frac{D}{2}+\frac{\Delta_j\left(R_2+\frac{D}{2}+\Delta_j\right)^2}{\left(R_2+\frac{D}{2}+\Delta_j\right)^2+z^2}\right] \tag{A30}$$

$$K_{2_j} = \frac{1}{\left\{\left(R_2+\frac{D}{2}+\Delta_j\right)^2+z^2\right\}^{1/2}} \cdot\left[\frac{z^2}{\left(R_2+\frac{D}{2}+\Delta_j\right)^2+z^2}\right] \tag{A31}$$

Appendix III
Approximate Coupling Coefficients

By use of the linear approximations similar to those of Appendix II and the isolated field equations, Appendix I, the coupling coefficients (7)–(10) for lossless, degenerate, but not necessarily symmetric, configurations can now be directly integrated with the following comments.

Equations (7)–(10) call for the vector fields in the x, y, z coordinate system. For the TE case, the following is easily shown:

$$\overline{E}_i = (0, E_y, 0) \tag{A32}$$

$$\overline{H}_1 = \left(H_{x_1}\cos\theta_1 + H_{z_1}\sin\theta_1, 0, -H_{x_1}\sin\theta_1 + H_{z_1}\cos\theta_1\right) \tag{A33}$$

$$\overline{H}_2 = \left(H_{x_2}\cos\theta_2 - H_{z_2}\sin\theta_2, 0, H_{x_2}\sin\theta_2 + H_{z_2}\cos\theta_2\right) \tag{A34}$$

where H_{x_1}, H_{x_2}, H_{z_1}, and H_{z_2} are part of the isolated field equations in Appendix I.

The integrals for A_1, A_2, B_1, and B_2 are only over guide 1 or guide 2 since

$$\tilde{\epsilon}(x,y,z)-\epsilon_{1,2}(x,y,z) = \begin{cases} 0, & \text{elsewhere} \\ (\epsilon_{2,1}-\epsilon_3), & \text{in guide } 2,1 \end{cases}. \tag{A35}$$

All of the coefficients have been divided by the half width of the guide a. All lengths are in terms of the guide half width

$$A_1 = \frac{(\epsilon_2-\epsilon_3)\cos^2(pa)\,e^{2\gamma a[1+S_{12}-(T_{12}/T_{22})S_{22}]}\sinh\left[2\gamma a\frac{T_{12}}{T_{22}}\right]}{\gamma a\cdot T_{12}} \tag{A36}$$

$$B_1 = (\epsilon_2 - \epsilon_3)\cos(pa)e^{\gamma a}e^{\gamma a[S_{12}-(T_{12}/T_{22})S_{22}]} \cdot e^{-j\beta a[v_2 - u_2(S_{22}/T_{22})]}\frac{2}{(paT_{22})^2+(\beta' a)^2}\left[\beta' a \sinh\left[\frac{\beta' a}{T_{22}}\right]\cos pa + pa\cdot T_{22}\cosh\left[\frac{\beta' a}{T_{22}}\right]\sin pa\right] \quad (A37)$$

where

$$\beta' a = \gamma a T_{12} - j\beta a U_2$$

$$C_1 = \frac{2\beta a}{(ka)^2}\left[1+\frac{1}{\gamma a}\right] \quad (A38)$$

$$D_1 = D_1' + D_1'' + D_1''' + D_1'''' \quad (A39)$$

$$D_1' = \sum_{i=1}^{4}\int_{A(i)}^{B(i)} \exp[G3(i)x][Q(i)+O(i)x]\,dx \quad (A40)$$

$$Q(i) = P(i)[\beta a(J_{1k}+J_{2k}) - \gamma a j[M(i)F_{1k}+N(i)F_{2k}] \quad (A41)$$

$$O(i) = P(i)[\beta a(K_{1k}+K_{2k}) - \gamma a j(M(i)G_{1k}+N(i)G_{2k})] \quad (A42)$$

$$P(i) = \frac{\cos(pa)}{ka^2}\exp[\beta a(2-N(i)\cdot S_{2k} - M(i)S_{1k} - j\beta a V_k] \quad (A43)$$

$$G3(i) = -\gamma a[M(i)T_{1k}+N(i)T_{2k}] - j\gamma a U_k \quad (A44)$$

$$M(1) = N(1) = N(2) = N(3) = 1 \quad (A45)$$

$$M(2) = M(3) = M(4) = N(4) = -1 \quad (A46)$$

$$B(1) = \infty; \quad A(1) = (1-S_{1k})/T_{1k}$$

$$B(2) = (1-S_{1k})/T_{1k}, \quad A(2) = 0$$

$$B(3) = 0; \quad A(3) = (1-S_{2k})/T_{2k}$$

$$B(4) = (-1-S_{2k})/T_{2k}, \quad A(4) = -\infty \quad (A47)$$

$$k = \begin{cases} 1, & i=1,2 \\ 2, & i=3,4 \end{cases} \quad (A48)$$

$$D_1'' = \sum_{i=1}^{2} Q(i)\int_{-1}^{1}\exp[G3(i)x]\sin[pax]\,dx \quad (A49)$$

$$G3(1) = \gamma a[T_{12}/T_{22} - j\beta a U_2/T_{22}] \quad (A50)$$

$$G3(2) = -\gamma a[T_{21}/T_{11} - j\beta a U_1/T_{22}] \quad (A51)$$

$$Q(1) = -\cos(pa)\exp(\gamma a)pa/ka^2(F_{22}+G_{22}\Delta_2)/T_{22} \cdot \exp[\gamma a(S_{12} - T_{12}\cdot S_{22}/T_{22}) - j\beta a(V_2 - U_2 S_{22}/T_{22})] \quad (A52)$$

$$Q(2) = \cos(pa)\exp(\gamma a)pa/ka^2(F_{11}+G_{11}\Delta_1)/T_{11} \cdot \exp[-a(S_{21} - T_{21}S_{11}/T_{11}) - j\beta a(V_1 - U_1 S_{11}/T_{11})] \quad (A53)$$

$$D_1''' = \frac{\beta a}{ka^2}\left[(J_{11}+J_{21}+\Delta_1(K_{11}+K_{21}))\frac{A_2^*}{\epsilon_1-\epsilon_3} + (J_{12}+J_{22}+\Delta_2(K_{12}+K_{22}))\frac{B_1}{\epsilon_2-\epsilon_3}\right\} \quad (A54)$$

$$D_1'''' = \frac{j\gamma a}{ka^2}\cdot\left[(F_{12}+\Delta_2 G_{12})\frac{B_1}{\epsilon_2-\epsilon_3} - (F_{21}+\Delta_1 G_{21})\frac{A_2^*}{\epsilon_1-\epsilon_3}\right] \quad (A55)$$

$$A_2 = (\epsilon_1-\epsilon_2)\cos(pa)e^{\gamma a}\exp\left[\gamma a\left[-S_{21}+\frac{T_{21}}{T_{11}}\cdot S_{11}\right]\right]\exp\left[-j\beta a\left[-V_1+\frac{S_{11}}{T_{11}}U_1\right]\right]\cdot\frac{2}{(paT_{11})^2+(\beta' a)^2}\cdot\left[\beta' a\sinh\left[\frac{\beta' a}{T_{11}}\right]\cos pa + paT_{11}\cosh\left[\frac{\beta' a}{T_{11}}\right]\sin pa\right] \quad (A56)$$

where

$$\beta' a = -\gamma a T_{21} + j\beta a U_1$$

$$B_2 = -(\epsilon_1-\epsilon_3)\frac{\cos^2 pa}{\gamma a T_{21}}\exp\left[2\gamma a\left(1 - S_{21} + \frac{T_{21}}{T_{11}}S_{11}\right)\right]\sinh\left[-2\gamma a\frac{T_{21}}{T_{11}}\right] \quad (A57)$$

$$C_2 = D_1^* \quad \text{(by inspection of (9) and (10))} \quad (A58)$$

$$D_2 = C_1 \quad \text{(by symmetry).} \quad (A59)$$

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Short Papers

Even- and Odd-Mode Impedances of Coupled Elliptic Arc Strips

B. N. DAS AND K. V. S. V. R. PRASAD

Abstract—**A derivation of the expression for even- and odd-mode impedances of coupled elliptic arc strips between grounded, confocal elliptic cylinders, and above a grounded elliptic cylinder, symmetrically located with the minor axis, is presented. The analysis is based on TEM-mode approximation. The Green's function formulation is used to obtain variational expressions for the even- and odd-mode capacitances for the more general case of different dielectrics on the two sides of the coupled strips. Numerical results are presented for coupled elliptic and circular cylindrical arc strips. It is also shown that the formulation can be used to find the effect of environmental changes on an otherwise planar structure.**

I. INTRODUCTION

Some investigations on elliptic and circular cylindrical striplines have been reported in the literature [1]–[4]. The impedance of warped lines can be determined from the results of the analysis of such lines by assuming that the radii of curvature are very large and the arc lengths remain finite. Wang [1] presented impedance data of cylindrical and cylindrically warped strip- and microstrip lines from numerical solution of Laplace's equation. He used the results of the analysis to calculate the effect of environmental changes on the impedance of an otherwise planar structure. The numerical results presented by him show a marked deviation from those calculated using the formula for planar structure. From physical considerations, however, it may be concluded that the impedance of a warped line should not differ appreciably from that of an otherwise planar structure. It has been established that the results obtained by Wang for warped lines are not correct [2], [4]. It is worthwhile to investigate the effect of environmental changes on the even- and odd-mode impedances of an otherwise planar structure. Expressions for these impedances can be obtained from the analysis of two coupled elliptic arc strips between confocal elliptic grounded cylinders. To the best of the author's knowledge, no investigation on coupled arc strips between grounded elliptic and circular cylinders or above such surfaces has been reported in the literature.

In the present work, a method of derivation of the expressions for the even- and odd-mode impedances of coupled elliptic and circular cylindrical arc strips between two dielectric layers is presented. If the transverse dimensions of the structure are small compared to the operating wavelength, quasi-TEM-mode approximation can be used for the analysis. The potential function for the even- and odd-mode configurations is derived using a Green's function formulation and TEM-mode approximation. Variational expressions for the even- and odd-mode capacitances are found, assuming suitable charge distribution on the arc strips. Formulation is made for the general case of elliptic arc strips between two confocal grounded elliptic cylinders. The corresponding expressions for the case of coupled elliptic arc strips above a grounded elliptic cylinder are found by assuming that the upper cylinder is moved to infinity. The even- and odd-mode impedances of coupled cylindrical strip- and microstripline are then found from the analysis of the elliptic line by assuming that eccentricity is equal to zero.

Numerical results on the even- and odd-mode impedances for 1) coupled elliptic arc strips between grounded elliptic cylinders

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